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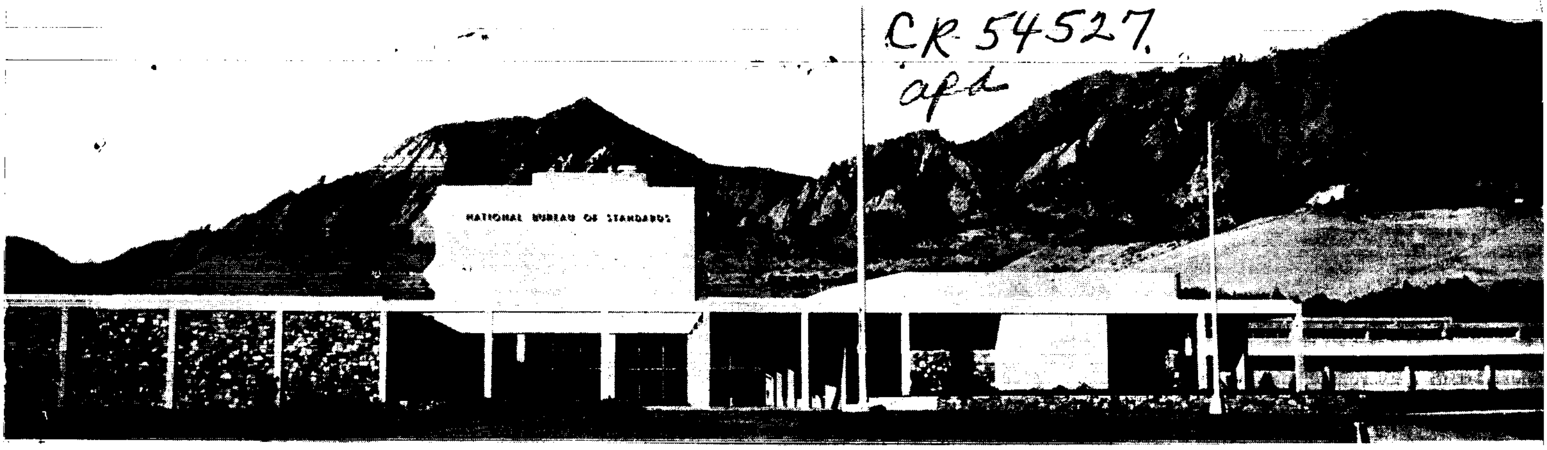

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by

R. W. Arnett and D. R. Millhiser

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FOREWORD

An analytical development is described that proposes a means for determining the extent and severity of thermal stratification of a side wall heated container and the associated self pressurization.

The work has been done as part of a consultation and advisory program supported by the National Aeronautics and Space Administration-Lewis Research Center, Centaur Project Office under Order Number C-69726.

A Method for Analyzing Thermal Stratification and Self Pressurization in a Fluid Container

R. W. Arnett and D. R. Millhiser

Introduction

The phenomena of thermal stratification in cryogenic liquids has been observed for several years [Huntley, 1960; Neff, 1960; Scott, et. al., 1960; Swim, 1960]. In recent years the importance of a better understanding of the causative mechanisms has been recognized and has resulted in numerous investigations of the problem being reported in the literature [Robbins and Rogers, 1964; Schmidt, et al., 1961; Schwind and Vliedt, 1964; Tatom, et al., 1964; Tellep and Harper, 1963].

Most of the referenced work has been oriented toward the thermally stratified layer as it affected liquid pumping due to reduced NPSH. Only a small amount of work is reported on the effect of thermal stratification on ullage pressure rise in a sealed container. Tellep and Harper [1963] have touched on this aspect but did not consider the effect of a possible temperature gradient in the ullage gas. The presence of a sizeable temperature change through the gas has been reported by Maxson [1963] and Schmidt, et al., [1961].

Coupling of the liquid surface to the gas space in such a manner as to allow prediction and measure the influence of condensation or vaporization has proven to be a difficult task. This presentation describes a means for predicting: (1) the extent of thermal stratification in the liquid, (2) the temperature gradients in the ullage space, and (3) the amount and direction of mass and energy interchange due to liquid-gas phase transformation at the liquid-gas interface.

Boundary Layer Equations

An analytical method for predicting the volume of liquid involved in thermal stratification and the temperature pattern existing in the stratified layer appears to be quite complicated. Many factors appear to influence the problem such as container geometry, heat transfer rate and location, fluid properties, force fields, and the variation with time of any or several of these factors. Further study and experimentation may reveal that the importance of some of these parameters is small or negligible, and simplified expressions may be used for satisfactory predictions. However, for this development the effect of factors as appear to be relevant will be retained although the development of the equations is approximate in some areas.

Since the immediate interest is the Centaur fuel tank the general shape of this tank is assumed, i. e., a cylindrical portion terminated at the top by a conical nose piece and closed at the bottom with a flat end. (See figure 1.) This latter assumption while not exact seems justifiable since it is certain that heat transfer through the bottom will be assimilated in the body of the fluid in either case.

The general method of attack follows that of Eckert and Jackson [1951] and Von Karman [1946] as modified for application to a cylindrical and conical shape. The flow pattern that appears from the approach used here is comprised of a free convection boundary layer forming adjacent to and flowing upward along the cylindrical walls. This layer proves to be turbulent in nature after only a short distance of travel from the tank bottom corner ($G^*_{R\chi} \geq 10^{11}$) for the Centaur conditions. Thermal energy passing through the wall is carried by this boundary layer through the bottom of the thermally stratified layer thus delivering the warmed fluid to the stratified layer. Since the stratified layer is a region where a temperature increase over the bulk temperature exists, it is quite apparent that thermal energy is being delivered to the fluid in this layer and this energy must come primarily from the boundary layer. If the boundary layer, as such disappeared at the lower boundary of the stratified layer, this would imply dumping of all the boundary layer energy at the bottom of the stratified layer. If such occurred, it is not apparent what mechanism would cause the distribution of this energy throughout the stratified layer with a non-negative temperature gradient. It seems reasonable that the boundary layer begins to lose mass as soon as it encounters the stratified layer and thus decays throughout its traverse of the warm layers of fluid. There is some uncertainty as to whether a complete decay occurs or if a finite boundary layer flow still exists at the surface causing "radially inward" flow to exist at the surface. The development reported here assumes complete decay of the boundary layer. Such a decay must be manifested by flow out from the boundary layer as it traverses the stratified layer. This flow would be composed of the coolest fluid in the layer namely, that fluid adjoining the inner face of the boundary layer. In this way the warmer fluid adjoining the wall would be delivered to the surface layers and cause the largest temperature increase to occur there, as in fact does seem to be the case.

Development of the expressions proceeds along lines similar to Von Karman [1946] and Eckert and Jackson, [1951] modified as required for a conical shape (see figure 2).

Following Eckert and Jackson, [1951] the temperature and velocity distribution in the boundary layer is assumed to be represented by

$$\theta = \theta_w\left[1-\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right]$$

and

$$u = U\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\left(1-\frac{y}{\delta}\right)^{4} .$$

Forces acting in the vertical direction on the element under consideration are made identically equal to the vertical change in momentum thru the element; Momentum change $\equiv \sum$ Forces (x),

The momentum through the lower plane is given by

$$2\pi \cos^2\gamma \int_0^\delta \rho u^2 (R-y)\,dy \;,$$

and the change in momentum in traversing the element is then

$$\left[2\pi \cos^2\gamma \frac{\partial}{\partial x}\int_0^\delta \rho u^2 (R-y)\,dy\right]dx \;. \tag{1}$$

Buoyant forces acting on the element are described by

$$\left[\int_0^\delta 2\pi g(\rho_B - \rho)(R-y)\,dy\right]dx \;, \tag{2}$$

directed upward. Setting

$$\rho_B - \rho = \rho\beta\theta$$

results in

$$2\pi g\beta \int_0^\delta \rho\theta (R-y)\,dy\,dx \;. \tag{2a}$$

Shear forces acting along the wall amount to

$$-2\pi R \tau_W\,dx \tag{3}$$

directed downward.

The argument is made that temperature changes are small and therefore there is little change in ρ from ρ_B and it may be assumed that $\rho/\rho_B \simeq 1$, except for arithmetic differences such as occurs in the buoyancy term. Here this difference is taken care of by introducing β, the volumetric coefficient of expansion.

Using the Blasius correlation for wall shear stress [Eckert and Drake, 1959, p. 143],

$$\tau_w = 0.0228 \rho U^2 \left(\frac{\nu}{U\delta \cos\gamma}\right)^{\frac{1}{4}} , \tag{4}$$

the expression for shear force becomes

$$-2\pi R \left[0.0228 \rho U^2 \left(\frac{\nu}{U\delta \cos\gamma}\right)^{\frac{1}{4}}\right] dx . \tag{3a}$$

When the indicated integrations are performed on equations (1) and (2a), they become

$$\left\{2\pi \rho_B \cos^2\gamma \frac{\partial}{\partial x}\left[U^2\delta\left(0.05232 R - 0.006539\delta\right)\right]\right\} dx \tag{1a}$$

and

$$2\pi \rho_B g \beta \left[\theta_w \delta \left(0.125 R - \frac{\delta}{30}\right)\right] dx . \tag{2b}$$

Equating the sum of forces to the change in momentum,

$$\begin{aligned} &2\pi \rho_B \cos^2\gamma \frac{\partial}{\partial x}\left[U^2\delta\left(0.05232 R - 0.006539\delta\right)\right] dx \\ &\equiv 2\pi \rho_B g \beta \theta_w \delta \left(0.125 R - \frac{\delta}{30}\right) dx \\ &-2\pi R (0.0228) \rho_B U^2 \left(\frac{\nu}{U\delta \cos\gamma}\right)^{\frac{1}{4}} dx . \end{aligned} \tag{5}$$

Simplifying

$$\begin{aligned} &\cos^2\gamma \frac{\partial}{\partial x}\left[0.05232 R U^2 \delta \left(1 - \frac{\delta}{8R}\right)\right] \\ &\equiv \tfrac{1}{8} g \beta R \theta_w \delta \left(1 - \frac{4\delta}{15R}\right) - 0.0228 R U^2 \left(\frac{\nu}{U\delta \cos\gamma}\right)^{\frac{1}{4}} . \end{aligned}$$

By combining the Reynolds analogy [Eckert and Drake, 1959, p. 203], relating viscous shear stress and conduction heat transfer, with the Blasius correlation and a correction term for Pr variation due to Colburn [Jakob, 1949; Eckert and Drake, 1959, p. 324] it is found that

$$\theta_W = \frac{g_W A^{\frac{2}{3}}}{0.0228\, c\, \rho_B U} \left(\frac{U \delta \cos\gamma}{\nu}\right)^{\frac{1}{4}} \qquad (4a)$$

Making this substitution the identity is now

$$\cos^2\gamma \frac{\partial}{\partial x}\left[0.05232\, R U^2 \delta \left(1 - \frac{\delta}{8R}\right)\right]$$

$$= \frac{g \beta g_W A^{\frac{2}{3}} \delta R}{0.1824 c\ \rho_B U}\left(\frac{U \delta \cos\gamma}{\nu}\right)^{\frac{1}{4}}\left(1 - \frac{4\delta}{15R}\right) \qquad (5a)$$

$$- 0.0228\, R U^2 \left(\frac{\nu}{U \delta \cos\gamma}\right)^{\frac{1}{4}} .$$

It is now assumed that U and δ have a similar mathematical form and are given by

$$U = C_1 x^m \ ,$$
$$\delta = C_2 x^n \ . \qquad (6)$$

Making this substitution on the left side of the identity produces

$$\cos^2\gamma \frac{\partial}{\partial x}\left[0.05232\, R C_1^2 C_2\, x^{2m+n}\left(1 - \frac{1}{8}\frac{C_2 x^n}{R}\right)\right] .$$

Performing the indicated differentiation, this becomes

$$0.05232\, C_1^2 C_2 (2m+n) R x^{2m+n-1}\left[1 + \frac{1}{2m+n}\frac{x}{R}\frac{\partial R}{\partial x} - \frac{C_2(2m+2n)}{8(2m+n)}\frac{x^n}{R}\right]\cos^2\gamma \ . \qquad (7)$$

Collecting all U and δ terms on the right side of the identity,

$$\frac{g \beta g_W A^{\frac{2}{3}} \delta^{\frac{5}{4}} R}{0.1824\, c\, \rho_B \nu^{\frac{1}{4}} U^{\frac{3}{4}}} \cos^{\frac{1}{4}}\gamma \left(1 - \frac{4}{15}\frac{\delta}{R}\right) - \frac{0.0228\, \nu^{\frac{1}{4}} R U^{\frac{7}{4}}}{\delta^{\frac{1}{4}} \cos^{\frac{1}{4}}\gamma} \ ,$$

and substituting for U and δ gives

$$\frac{g \beta g_W A^{\frac{2}{3}}}{0.1824 c\ \rho_B \nu^{\frac{1}{4}}} C_2^{\frac{5}{4}} C_1^{-\frac{3}{4}} x^{\frac{5n-3m}{4}} R \cos^{\frac{1}{4}}\gamma \left(1 - C_2 \frac{4}{15}\frac{x^n}{R}\right) \qquad (8)$$

$$- 0.0228 \frac{\nu^{\frac{1}{4}}}{\cos^{\frac{1}{4}}\gamma}\ C_1^{\frac{7}{4}} C_2^{-\frac{1}{4}} R x^{\frac{7m-n}{4}} .$$

Combining expressions (7) and (8) gives

$$0.05232\, C_1^2 C_2 (2m+n) R x^{2m+n-1} \left[1 + \frac{1}{2m+n} \frac{x}{R} \frac{\partial R}{\partial x} - \frac{C_2 (2m+2n)}{8(2m+n)} \frac{x^n}{R} \right] \cos^2 \gamma$$

$$= \frac{g \beta q_w A^{\frac{2}{3}}}{0.1824 c\, \rho_B \nu^{\frac{1}{4}}} C_2^{\frac{5}{4}} C_1^{-\frac{3}{4}} x^{\frac{5n-3m}{4}} R \cos^{\frac{1}{4}} \gamma \left(1 - C_2 \frac{4}{15} \frac{x^n}{R} \right) \qquad (5b)$$

$$- 0.0228 \frac{\nu^{\frac{1}{4}}}{\cos^{\frac{1}{4}} \gamma} C_1^{\frac{7}{4}} C_2^{-\frac{1}{4}} R x^{\frac{7m-n}{4}} .$$

For a cylindrical shape $\partial R/\partial x = 0$, and the middle bracketed term on the left side disappears. For an infinite flat plate $R \rightarrow \infty$, and the term involving x^n/R disappears leaving an equation similar to that developed by other investigators [Morse, 1962].

In order that the above identity holds for all values of x in the range of interest the exponents of x must be equal, thus the following conditions must hold:

$$\begin{aligned} 2m+n-1 &= \frac{5n-3m}{4} = \frac{7m-n}{4} \\ 2m+2n-1 &= \frac{9n-3m}{4} . \end{aligned} \qquad (9)$$

Solution of the first set of equations leads to $m = \frac{3}{7}$, $n = \frac{5}{7}$, which turns out to be consistent with the second set of equations. The first set of equations would be valid for a flat vertical plate while both the first and second must hold for a cylindrical shape. A conical shape calls for R to be a linear function of x, and thus $R = R_C - x \tan \gamma$ and $\frac{\partial R}{\partial x} = -\tan \gamma$.

When the term involving $\partial R/\partial x$ is retained, a third equation emerges which has no direct solution and is inconsistent with the first two, i.e., $2m+n \rightarrow -\infty$. It appears that with the conical shape the assumption of constant values for m and n may not be valid; rather they would appear to vary with the cone half angle and perhaps with vertical location.

Due to the complexity of obtaining a solution giving values for m and n to suit this last condition coupled with the relatively short time element involved for the stratified layer growth in the nose cone additional time was not spent on this phase. Thus expressions are obtained for the boundary layer consisting of

$$U = C_1 x^{\frac{3}{7}}$$

and (6a)

$$\delta = C_2 x^{\frac{5}{7}} .$$

Determination of the values of C_1 and C_2 is next accomplished.

Thermal energy entering the bottom of the element under consideration is given by

$$\int_0^\delta 2\pi c\,\theta\,\rho\,u\,(R-y)\,dy\cos\gamma .$$

The reference datum used here is the initial bulk fluid condition. Change in thermal energy through the element is then

$$\left[\frac{\partial}{\partial x}\int_0^\delta 2\pi c\,\theta\,\rho\,u\,(R-y)\,dy\cos\gamma\right]dx . \qquad (10)$$

Energy entering through the wall is given as

$$2\pi R q_W \frac{dx}{\cos\gamma} . \qquad (11)$$

Fluid entering or leaving through the inner face of the boundary layer enters at the datum condition and therefore has zero energy. Utilizing the assumption $\rho/\rho_B \simeq 1$, the energy equation then becomes

$$\left[2\pi c\,\rho_B\cos\gamma\frac{\partial}{\partial x}\int_0^\delta \theta\,u\,(R-y)\,dy\right]dx \equiv 2\pi R q_W \frac{dx}{\cos\gamma} . \qquad (12)$$

Performing the integration and simplifying:

$$c\rho_B\cos^2\gamma\frac{\partial}{\partial x}\left[0.03663 R\theta_W U\delta - 0.004785\,\theta_W U\delta^2\right] \equiv q_W R , \qquad (12a)$$

or

$$c/\rho_B \cos^2\gamma \frac{\partial}{\partial x}\left[R\theta_W U\delta\left(0.03663 - 0.004785\,\frac{\delta}{R}\right)\right] \equiv g_W R\ .$$

Substituting for θ_W from equation (4b) results in

$$c/\rho_B \cos^2\gamma \frac{\partial}{\partial x}\left\{\frac{g_W Pr^{\frac{2}{3}} R U\delta}{0.0228\, c\rho_B U}\left(\frac{U\delta\cos\gamma}{\nu}\right)^{\frac{1}{4}}\left(0.03663 - 0.004785\,\frac{\delta}{R}\right)\right\} \equiv g_W R\ , \quad (12b)$$

or

$$\frac{g_W Pr^{\frac{2}{3}}\cos^{\frac{9}{4}}\gamma}{0.0228\,\nu^{\frac{1}{4}}}\,\frac{\partial}{\partial x}\left\{R\,U^{\frac{1}{4}}\delta^{\frac{5}{4}}\left(0.03663 - 0.004785\,\frac{\delta}{R}\right)\right\} \equiv g_W R.$$

Further substiuting for U and δ from (6) produces

$$\left[\frac{Pr^{\frac{2}{3}}\cos^{\frac{9}{4}}\gamma}{0.0228\,\nu^{\frac{1}{4}}R}\right]\frac{\partial}{\partial x}\left\{C_1^{\frac{1}{4}}C_2^{\frac{5}{4}}x^{\frac{m+5n}{4}}\left[0.03663R - 0.004785\,C_2\,x^n\right]\right\} \equiv 1\ .$$

Performing the differentiation:

$$\left[\frac{Pr^{\frac{2}{3}}\cos^{\frac{9}{4}}\gamma}{0.0228\,\nu^{\frac{1}{4}}R}\right]\left\{C_1^{\frac{1}{4}}C_2^{\frac{5}{4}}x^{\frac{m+5n}{4}}\left[0.03663\,\frac{\partial R}{\partial x} - 0.004785\,C_2\,n\,x^{n-1}\right]\right.$$
$$\left. + C_1^{\frac{1}{4}}C_2^{\frac{5}{4}}\left[0.03663R - 0.004785\,C_2\,x^n\right]\left(\frac{m+5n}{4}\right)x^{\frac{m+5n-4}{4}}\right\} \equiv 1\ .$$

This reduces to

$$\frac{1.6066\,C_1^{\frac{1}{4}}C_2^{\frac{5}{4}}Pr^{\frac{2}{3}}\cos^{\frac{9}{4}}\gamma}{\nu^{\frac{1}{4}}}\,x^{\frac{m+5n}{4}}\left\{\left[\frac{1}{R}\frac{\partial R}{\partial x} - 0.1306\,C_2\,n\,\frac{x^{n-1}}{R}\right]\right.$$
$$\left. + \left(\frac{m+5n}{4}\right)\left(\frac{1}{x}\right)\left[1 - 0.1306\,C_2\,\frac{x^n}{R}\right]\right\} \equiv 1\ . \quad (12c)$$

Two equations for exponential values can be obtained from this as was possible with the previous equation. These are

$$\frac{m+5n-4}{4} = 0 \qquad \text{and} \qquad \frac{m+9n-4}{4} \rightarrow -\infty\ . \quad (9a)$$

The values previously obtained of $m = \frac{3}{7}$, $n = \frac{5}{7}$ satisfy the first equation but are obviously not a solution to the second equation which stems from the conical shape. Because of previously mentioned arguments this last equation was not considered further.

When the thermal energy equation is stated with the above values for m and n inserted, the following expression results:

$$1.6066\,\frac{C_1^{\frac{1}{4}}C_2^{\frac{5}{4}}Pr^{\frac{2}{3}}\cos^{\frac{9}{4}}\gamma}{\nu^{\frac{1}{4}}}\left[\frac{x}{R}\frac{\partial R}{\partial x}-0.2239\,C_2\frac{x^n}{R}+1\right]\equiv 1\,. \qquad (12d)$$

Solving for C_1,

$$C_1=\left[\frac{\nu}{6.6628\,Pr^{\frac{8}{3}}\cos^9\gamma\,C_2^5}\right]\left[\frac{1}{1+\frac{x}{R}\frac{\partial R}{\partial x}-0.2239\,C_2\frac{x^n}{R}}\right]^4. \qquad (13)$$

Returning to the momentum equation and inserting values for m and n, results in

$$(0.08221)\;C_1^2C_2\left[1+\frac{7}{11}\frac{x}{R}\frac{\partial R}{\partial x}-\frac{2}{11}C_2\frac{x^n}{R}\right]\cos^2\gamma$$

$$\equiv 5.4825\,\frac{g\beta q_w Pr^{\frac{2}{3}}C_2^{\frac{5}{4}}}{c\rho_B\nu^{\frac{1}{4}}C_1^{\frac{1}{4}}}\left[1-\frac{4}{15}C_2\frac{x^n}{R}\right]\cos^{\frac{1}{4}}\gamma-0.0228\,\frac{\nu^{\frac{1}{4}}}{\cos^{\frac{1}{4}}\gamma}\,\frac{C_1^{\frac{7}{4}}}{C_2^{\frac{1}{4}}}\,.$$

Substituting for C_1 from (13) gives

$$\frac{0.08221\,\nu^2}{(6.6628)^2\,Pr^{\frac{16}{3}}\cos^{16}\gamma\,C_2^9}\left[1+\frac{7}{11}\frac{x}{R}\frac{\partial R}{\partial x}-0.1818\,C_2\frac{x^n}{R}\right]\left[\frac{1}{1+\frac{x}{R}\frac{\partial R}{\partial x}-0.2239\,C_2\frac{x^n}{R}}\right]^8$$

$$=22.736\,\frac{g\beta q_w Pr^{\frac{8}{3}}C_2^5}{c\rho_B\,\nu}\cos^7\gamma\left[1-0.2667\,C_2\frac{x^n}{R}\right]\left[1+\frac{x}{R}\frac{\partial R}{\partial x}-0.2239\,C_2\frac{x^n}{R}\right]^3$$

$$-\frac{0.0228\,\nu^2}{(6.6628)^{\frac{7}{4}}Pr^{\frac{14}{3}}C_2^9\cos^{16}\gamma}\left[\frac{1}{1+\frac{x}{R}\frac{\partial R}{\partial x}-0.2239\,C_2\frac{x^n}{R}}\right]^7,$$

and

$$C_2^{14}=\frac{c\rho_B\,\nu^3}{22.736\,g\beta q_w Pr^8\cos^{23}\gamma}\left[\frac{1}{1-0.2667\,C_2\frac{x^n}{R}}\right]\left[\frac{1}{1+\frac{x}{R}\frac{\partial R}{\partial x}-0.2239\,C_2\frac{x^n}{R}}\right]^{11} \qquad (14)$$

$$\left\{\frac{1}{540}\left[1+\frac{7}{11}\frac{x}{R}\frac{\partial R}{\partial x}-0.1818\,C_2\frac{x^n}{R}\right]+\frac{Pr^{\frac{2}{3}}}{1211.9}\left[1+\frac{x}{R}\frac{\partial R}{\partial x}-0.2239\,C_2\frac{x^n}{R}\right]\right\}.$$

For ease of handling the following substitutions are made:

$$
\begin{aligned}
K_1 &= 22.736 \\
K_2 &= 1 - 0.2667\, C_2 \frac{x^n}{R} \\
K_3 &= 540 \\
K_4 &= 1 + \frac{7}{11}\frac{x}{R}\frac{\partial R}{\partial x} - 0.1818\, C_2 \frac{x^n}{R} \\
K_5 &= 1211.9 \\
K_6 &= 1 + \frac{x}{R}\frac{\partial R}{\partial x} - 0.2239\, C_2 \frac{x^n}{R} \\
K_7 &= 6.6628\,.
\end{aligned} \tag{15}
$$

This now gives

$$C_2^{14} = \frac{1}{K_1 K_2 K_6^{11}}\,\frac{c\rho_B \nu^3}{g\beta q_w Pr^8 \cos^{23}\gamma}\left[\frac{K_4}{K_3} + Pr^{\frac{2}{3}}\frac{K_6}{K_5}\right].$$

Therefore,

$$C_2 = \left[\frac{1}{K_1 K_2 K_6^{11}}\right]^{\frac{1}{14}}\left[\frac{c\,\rho_B \nu^3}{g\beta q_w Pr^8 \cos^{23}\gamma}\right]^{\frac{1}{14}}\left[\frac{K_4}{K_3} + Pr^{\frac{2}{3}}\frac{K_6}{K_5}\right]^{\frac{1}{14}}, \tag{14a}$$

and

$$C_1 = \frac{1}{K_7 K_6^4}\left[\frac{\nu}{Pr^{\frac{8}{3}}\cos^9\gamma\; C_2^5}\right]$$

$$= \left[\frac{1}{K_7 K_6^4}\right]\left[\frac{\nu}{Pr^{\frac{8}{3}}\cos^9\gamma}\right]\left[K_1 K_2 K_6^{11}\right]^{\frac{5}{14}}\left[\frac{K_4}{K_3} + Pr^{\frac{2}{3}}\frac{K_6}{K_5}\right]^{-\frac{5}{14}}\left[\frac{g\beta q_w Pr^8 \cos^{23}\gamma}{c\rho_B \nu^3}\right]^{\frac{5}{14}}.$$

Thus,

$$C_1 = \frac{[K_1 K_2]^{\frac{5}{14}}}{K_7 K_6^{\frac{1}{14}}}\left[\frac{K_4}{K_3} + Pr^{\frac{2}{3}}\frac{K_6}{K_5}\right]^{-\frac{5}{14}}\left[\frac{g\beta q_w}{c\,\rho_B}\right]^{\frac{5}{14}}\left[\frac{Pr^{\frac{4}{21}}}{\nu^{\frac{1}{14}}\cos^{\frac{11}{14}}\gamma}\right]. \tag{13a}$$

For the case of a vertical flat plate the terms K_2, K_4, K_6, and $\cos\gamma$ all reduce to 1. Which then results in

$$C_2 = \left(\frac{1}{K_1}\right)^{\frac{1}{14}}\left[\frac{c\rho_B \nu^3}{g\beta q_w Pr^8}\right]^{\frac{1}{14}}\left[\frac{1}{K_3} + Pr^{\frac{2}{3}}\frac{1}{K_5}\right]^{\frac{1}{14}},$$

and

$$C_1 = \left(\frac{K_1^{\frac{5}{14}}}{K_7}\right)\left[\frac{1}{K_3} + Pr^{\frac{2}{3}}\frac{1}{K_5}\right]^{-\frac{5}{14}}\left[\frac{g\beta q_w}{c\rho_B}\right]^{\frac{5}{14}}\left[\frac{Pr^{\frac{4}{21}}}{\nu^{\frac{1}{14}}}\right].$$

Simplifying,

$$C_2 = \left(\frac{1}{K_1K_5}\right)^{\frac{1}{14}}\left[\frac{c\rho_B\nu^3}{g\beta q_w Pr^8}\right]^{\frac{1}{14}}\left[\frac{K_5}{K_3} + Pr^{\frac{2}{3}}\right]^{\frac{1}{14}},$$

$$C_1 = \frac{[K_1K_5]^{\frac{5}{14}}}{K_7}\left[\frac{g\beta q_w}{c\rho_B}\right]^{\frac{5}{14}}\left[\frac{Pr^{\frac{4}{21}}}{\nu^{\frac{1}{14}}}\right]\left[\frac{K_5}{K_3} + Pr^{\frac{2}{3}}\right]^{-\frac{5}{14}},$$

and finally,

$$C_2 = 0.4818\left[\frac{c\rho_B\nu^3}{g\beta q_w Pr^8}\right]^{\frac{1}{14}}\left[2.2442 + Pr^{\frac{2}{3}}\right]^{\frac{1}{14}}, \qquad (14b)$$

$$C_1 = 5.7826\left[\frac{g\beta q_w}{c\rho_B}\right]^{\frac{5}{14}}\left[\frac{Pr^{\frac{4}{21}}}{\nu^{\frac{1}{14}}}\right]\left[2.2442 + Pr^{\frac{2}{3}}\right]^{-\frac{5}{14}}, \qquad (13b)$$

or

$$C_2 = 0.4818\ \frac{L^{\frac{2}{7}}}{Gr_L^{\frac{1}{14}}Pr^{\frac{1}{2}}}\left[2.2442 + Pr^{\frac{2}{3}}\right]^{\frac{1}{14}}, \qquad (14c)$$

$$C_1 = 5.7826\ \frac{Gr_L^{\frac{5}{14}}\nu}{Pr^{\frac{1}{6}}L^{\frac{10}{7}}}\left[2.2442 + Pr^{\frac{2}{3}}\right]^{-\frac{5}{14}}. \qquad (13c)$$

This compares closely with other investigators' results for a vertical flat plate when evaluated for $Pr = 1$.

For the case of either a cylinder with constant finite radius or a cone with a varying finite radius an iteration procedure can be used to evaluate C_1 and C_2. This is accomplished by assuming a value for C_2

(a first approximation would be from the equations for a vertical flat plate), calculating the values for K_2, K_4, and K_6, evaluating C_2, and comparing with the assumed value. This procedure would need to be repeated for the various values of x. As an example the constant C_2 has a value of approximately 0.35 at the liquid surface in the Centaur LH_2 tank for the conditions of $q_W = 0.02\ W/cm^2$, $P = 1.5\ atm.$ and $1g$ acceleration. For the same conditions on a vertical flat plate, C_2 has a value of approximately 0.25. Only slight variations in the value of C_2 and C_1 are experienced with moderate changes in heat flux and ullage pressure.

From the above it is possible to predict the velocity and thickness of the free convection boundary layer at any point, using either iteration for determining the constants or an approximation for the mean value.

Using these equations it is seen that a different constant is determined for the cylindrical portion than for the conical portion. This difference is partly due to the choice of measuring thickness perpendicular to the tank axis while velocity is measured parallel to the tank wall. An effect due to the sloping surface is also present regardless of the coordinate system chosen. Because of the differences mentioned above a correction term must be applied to the boundary layer equations in order to assure continuity of mass flow and boundary layer thickness at the cylinder-cone transition. Using subscript n to designate conical nose section, the expressions are written

$$\begin{aligned} \delta_n &= C_{2n} x^{\frac{5}{7}} + F_2 , \quad L_c < x \le L \\ \delta_c &= C_{2c} x^{\frac{5}{7}} , \quad 0 < x \le L_c , \end{aligned} \qquad (16)$$

where the subscript C designates the cylindrical portion of the tank. To determine the value of F_2 it is necessary that the boundary layer thickness perpendicular to the wall be the same regardless of which equation is used at the limiting value of $x = L_C$. Thus

$$\left[\delta_n \cos\gamma = \delta_C\right]_{x=L_C} ,$$

and

$$C_{2n} L_C^{\frac{5}{7}} \cos\gamma + F_2 \cos\gamma = C_{2C} L_C^{\frac{5}{7}} .$$

Therefore

$$F_2 = \frac{C_{2C}}{\cos\gamma} L_C^{\frac{5}{7}} - C_{2n} L_C^{\frac{5}{7}} = L_C^{\frac{5}{7}}\left(\frac{C_{2C}}{\cos\gamma} - C_{2n}\right) .$$

Likewise it is required that U have the same value at $x = L_C$, and it is observed that

$$U_n = C_{1n} x^{\frac{3}{7}} + F_1 \quad , \quad L_C \leq x \leq L \tag{17}$$
$$U_C = C_{1C} x^{\frac{3}{7}} \quad , \quad 0 < x \leq L_C ,$$

and that

$$\left[U_n = U_C\right]_{x=L_C} .$$

Therefore

$$C_{1n} L_C^{\frac{3}{7}} + F_1 = C_{1C} L_C^{\frac{3}{7}} ,$$

and

$$F_1 = L_C^{\frac{3}{7}} \left(C_{1C} - C_{1n}\right) .$$

Boundary layer parameters of thickness and velocity can then be stated for both the cylindrical and conical portion of the tank and still preserve the continuity of mass flow.

Growth of the Stratified Layer

Thickness of the stratified layer will be designated by Δ . An expression relating boundary layer flow to growth rate of the stratified layer is given as

$$A_\delta\left[\frac{1}{A_\delta}\frac{\partial V_\delta}{\partial t}-\frac{\partial \Delta}{\partial t}\right]=A_\Delta\frac{\partial \Delta}{\partial t} \tag{18}$$

where $\partial V_\delta/\partial t$ is the volume flow rate in the boundary layer at the bottom of the stratified layer, A_δ is the cross-sectional area of the boundary layer, and A_Δ is the entire tank cross-sectional area at the same location. Expressions for the above are given as

$$\left(\frac{\partial V_\delta}{\partial t}\right)_\Delta = 2\pi R_\Delta U_\Delta \delta_\Delta \cos\gamma\left[0.1464 - 0.02723\,\frac{\delta_\Delta}{R_\Delta}\right],$$

$$A_\delta = 2\pi\left(R_\Delta - \frac{\delta_\Delta}{2}\right)\delta_\Delta ,$$

and

$$A_\Delta = \pi R_\Delta^2 .$$

Introducting these into (18) and rearranging, yields

$$\frac{\partial \Delta}{\partial t} = \frac{1}{A_\Delta - A_\delta}\left(\frac{\partial V_\delta}{\partial t}\right)_\Delta$$

$$= \frac{2\pi R_\Delta U_\Delta \delta_\Delta \cos\gamma}{\pi R_\Delta^2 - 2\pi\left(R_\Delta - \frac{\delta_\Delta}{2}\right)\delta_\Delta}\left[0.1464 - 0.02723\,\frac{\delta_\Delta}{R_\Delta}\right] \tag{18a}$$

$$= \frac{2 U_\Delta \delta_\Delta \cos\gamma}{R_\Delta - \delta_\Delta\left(2 - \frac{\delta_\Delta}{R_\Delta}\right)}\left[0.1464 - 0.02723\,\frac{\delta_\Delta}{R_\Delta}\right] .$$

An explicit integration is difficult here since U_Δ , δ_Δ , and R_Δ are or may be functions of Δ . However a numerical integration will permit determination of Δ as a function of time with sufficient accuracy.

Stratified Layer Temperature Gradient

The model employed here uses a decaying free convection boundary layer with an initial mass flow at the stratified layer lower surface equal to that entering from the boundary layer existing in the lower regions of the tank (see figure 3).

Temperature and velocity distribution through the boundary layer are assumed to be similar to those used previously for the boundary layer, i. e.,

$$\theta_\ell = \theta_{w\ell}\left[1-\left(\frac{y}{\delta_\ell}\right)^{\frac{1}{7}}\right] ,$$

and

$$u_\ell = U_\ell\left(\frac{y}{\delta_\ell}\right)^{\frac{1}{7}}\left(1-\frac{y}{\delta_\ell}\right)^4 .$$

Bulk liquid at temperature T_B is considered to be the datum condition.

The decay of the boundary layer as it traverses the stratified layer (see figure 3) is forced by assuming the relations

$$\delta_\ell = \delta_\Delta\left(1-\frac{z}{\Delta}\right)^r , \qquad (19)$$

and

$$U_\ell = U_\Delta\left(1-\frac{z}{\Delta}\right)^p .$$

A development parallel to that made previously is used.

Momentum entering the element is

$$\int_0^{\delta_\ell} 2\pi\rho(R-y)\,u_\ell^2\,dy\,\cos^2\gamma ,$$

and the change in momentum is

$$\frac{\partial}{\partial z}\left[\int_0^{\delta_\ell} 2\pi\rho(R-y)\,u_\ell^2\,dy\,\cos^2\gamma\right]dz . \qquad (1b)$$

Buoyant force acting upward is

$$\int_0^{\delta_\ell} g(\rho_\ell - \rho) 2\pi (R-y) dy\, dZ$$
$$= \int_0^{\delta_\ell} g \rho \beta \theta_\ell 2\pi (R-y) dy\, dZ . \qquad (2c)$$

Shear force acting downward is

$$-2\pi R \tau_W \frac{dZ}{\cos\gamma} \cos\gamma = -2\pi R \tau_W dZ . \qquad (3b)$$

Equating 1b to the sum of 2c and 3b results in

$$\frac{\partial}{\partial Z}\left[2\pi \rho_B \cos^2\gamma \int_0^{\delta_\ell} (R-y)\, u_\ell^2\, dy\right] dZ$$
$$= 2\pi g \rho_B \beta \int_0^{\delta_\ell} (R-y) \theta_\ell\, dy\, dZ - 2\pi R \tau_W dZ . \qquad (5c)$$

Evaluating integrals and dividing out constant terms,

$$\frac{\partial}{\partial Z}\left[\delta_\ell U_\ell^2 (0.05232 R - 0.006539\, \delta_\ell) \cos^2\gamma\right]$$
$$\equiv g \beta \delta_\ell \theta_{W\ell} (0.125 R - \tfrac{1}{30} \delta_\ell) - \frac{R}{\rho_B} \tau_W .$$

Substituting for $\theta_{W\ell}$ and τ_W from the Blasius correlation and Reynold's analogy, produces

$$\frac{\partial}{\partial Z}\left[\delta_\ell U_\ell^2 (0.05232 R - 0.006539\, \delta_\ell) \cos^2\gamma\right]$$
$$\equiv \frac{g \beta q_W P_r^{\frac{2}{3}} \delta_\ell^{\frac{5}{4}}}{0.0228\, c \rho_B \nu^{\frac{1}{4}} U_\ell^{\frac{3}{4}}} (0.125 R - \tfrac{1}{30} \delta_\ell) \cos^{\frac{1}{4}}\gamma - \frac{0.0228 R \rho_B \nu^{\frac{1}{4}} U_\ell^{\frac{7}{4}}}{\delta_\ell^{\frac{1}{4}} \cos^{\frac{1}{4}}\gamma} .$$

Substituting for U_ℓ and δ_ℓ from (19),

$$\frac{\partial}{\partial Z}\left\{\delta_\Delta U_\Delta^2 \left(1 - \frac{Z}{\Delta}\right)^{2p+r} \left[0.05232 R - 0.006539\, \delta_\Delta \left(1 - \frac{Z}{\Delta}\right)^r\right] \cos^2\gamma\right.$$
$$\equiv \frac{g \beta q_W P_r^{\frac{2}{3}} \delta_\Delta^{\frac{5}{4}}}{0.0228\, c\, \nu^{\frac{1}{4}} U_\Delta^{\frac{3}{4}}} \left(1 - \frac{Z}{\Delta}\right)^{\frac{5r-3p}{4}} \left[0.125 R - \tfrac{1}{30} \delta_\Delta \left(1 - \frac{Z}{\Delta}\right)^r\right] \cos^{\frac{1}{4}}\gamma$$
$$- \frac{0.0228 R \rho_{B\ell} \nu^{\frac{1}{4}} U_\Delta^{\frac{7}{4}}}{\delta_\Delta^{\frac{1}{4}} \cos^{\frac{1}{4}}\gamma} \left(1 - \frac{Z}{\Delta}\right)^{\frac{7p-r}{4}} .$$

After performing the differentiation and simplifying this becomes,

$$\frac{\delta_\Delta U_\Delta^2}{\Delta}\cos^2\gamma\left(1-\frac{Z}{\Delta}\right)^{2p+r-1}\left\{0.05232\left(1-\frac{Z}{\Delta}\right)\frac{\Delta}{R}\frac{\partial R}{\partial Z}+0.006539(2p+2r)\frac{\delta_\Delta}{R}\left(1+\frac{Z}{\Delta}\right)^r\right.$$

$$\left.-0.05232(2p+r)\right\}\equiv\frac{g\beta q_w R^{\frac{2}{3}}\delta_\Delta^{\frac{5}{4}}}{0.0228\,c\,\nu^{\frac{1}{4}}U_\Delta^{\frac{7}{4}}}\left(1-\frac{Z}{\Delta}\right)^{\frac{5r-3p}{4}}\left[0.125-\frac{1}{30}\frac{\delta_\Delta}{R}\left(1-\frac{Z}{\Delta}\right)^r\right]\cos^{\frac{1}{4}}\gamma \tag{5d}$$

$$-\frac{0.0228\,\rho_B\,\nu^{\frac{1}{4}}U_\Delta^{\frac{7}{4}}}{\delta_\Delta^{\frac{1}{4}}\cos^{\frac{1}{4}}\gamma}\left(1-\frac{Z}{\Delta}\right)^{\frac{7p-r}{4}} .$$

This equation is exactly analogous to the one developed earlier for the growing boundary layer, and matching values for p and r are obtained, i. e.,

$$p=\frac{3}{7}\ ,\ r=\frac{5}{7}\ .$$

This produces

$$\delta_\ell=\delta_\Delta\left(1-\frac{Z}{\Delta}\right)^{\frac{5}{7}}\ ,\ U_\ell=U_\Delta\left(1-\frac{Z}{\Delta}\right)^{\frac{3}{7}}\ .$$

Turning now to a thermal energy balance, the thermal energy entering the element is

$$\int_0^{\delta_\ell}2\pi c\rho(\theta_\ell+\theta_{B\ell})(R-y)\,u_\ell\cos\gamma\,dy\ ,$$

and the change in energy through the elment is

$$\frac{\partial}{\partial Z}\left[\int_0^{\delta_\ell}2\pi c\rho(\theta_\ell+\theta_{B\ell})(R-y)\,u_\ell\cos\gamma\,dy\right]dZ\ . \tag{10a}$$

Energy input through the heated wall is

$$\frac{2\pi R q_w}{\cos\gamma}\,dZ\ . \tag{11a}$$

The energy flow across the inner face is dependent upon the mass flow across this face which in turn is equal to the change in mass flow crossing the element. Entering the lower face there is a mass flow of

$$\int_0^{\delta_\ell}2\pi\rho u_\ell(R-y)\cos\gamma\,dy\ ,$$

and the change in mass flow is

$$\frac{\partial}{\partial Z}\left[\int_0^{\delta_\ell} 2\pi \rho u_\ell (R-y)\cos\gamma\, dy\right] dZ \, . \tag{20}$$

Energy then enters in the amount

$$c\,\theta_{B\ell}\left\{\frac{\partial}{\partial Z}\left[\int_0^{\delta_\ell} 2\pi \rho u_\ell (R-y)\cos\gamma\, dy\right] dZ\right\} . \tag{20a}$$

Equating energy change to energy entering the element gives

$$\frac{\partial}{\partial Z}\left[\int_0^{\delta_\ell} 2\pi c \rho (\theta_\ell + \theta_{B\ell})(R-y) u_\ell \cos\gamma\, dy\right] dZ$$

$$\equiv \frac{2\pi R q_w}{\cos\gamma}\, dZ + c\,\theta_{B\ell}\left\{\frac{\partial}{\partial Z}\left[\int_0^{\delta_\ell} 2\pi \rho u_\ell (R-y)\cos\gamma\, dy\right] dZ\right\} . \tag{12e}$$

The energy movement across the inner face of the element is included here since mass crossing this boundary in the stratified layer does so at a temperature other than the bulk temperature, thus carrying energy with it.

Simplifying this expression produces

$$\frac{\partial}{\partial Z}\left[R\int_0^{\delta_\ell} \theta_\ell u_\ell\, dy - \int_0^{\delta_\ell} \theta_\ell u_\ell y\, dy + \theta_{B\ell} R\int_0^{\delta_\ell} u_\ell\, dy - \theta_{B\ell}\int_0^{\delta_\ell} u_\ell y\, dy\right]$$

$$\equiv \frac{R q_w}{c \rho_B \cos^2\gamma} + \theta_{B\ell}\left\{\frac{\partial}{\partial Z}\left[R\int_0^{\delta_\ell} u_\ell\, dy - \int_0^{\delta_\ell} u_\ell y\, dy\right]\right\} .$$

Evaluating integrals gives

$$\frac{\partial}{\partial Z}\Big[0.03663\, R\,\theta_{w\ell} U_\ell \delta_\ell - 0.004785\, \theta_{w\ell} U_\ell \delta_\ell^2$$

$$+ 0.1464\, \theta_{B\ell} R U_\ell \delta_\ell - 0.02723\, \theta_{B\ell} U_\ell \delta_\ell^2\Big]$$

$$\equiv \frac{R q_w}{c \rho_B \cos^2\gamma} + \theta_{B\ell}\frac{\partial}{\partial Z}\left[0.1464\, R U_\ell \delta_\ell - 0.02723\, U_\ell \delta_\ell^2\right] .$$

Substituting for $\theta_{W\ell}$ from the Blasius-Reynolds analogy correlation (equation 4a) produces

$$\left[\frac{g_W A^{\frac{2}{3}} \cos^{\frac{1}{4}}\gamma}{0.0228\, c\, \rho_B\, \nu^{\frac{1}{4}}}\right]\frac{\partial}{\partial z}\left[0.03663\, R\, U_\ell^{\frac{1}{4}}\delta_\ell^{\frac{5}{4}} - 0.004785\, U_\ell^{\frac{1}{4}}\delta_\ell^{\frac{9}{4}}\right]$$

$$+ \frac{\partial}{\partial z}\left[0.1464\, R\,\theta_{B\ell} U_\ell \delta_\ell - 0.02723\,\theta_{B\ell} U_\ell \delta_\ell^{2}\right]$$

$$\equiv \frac{R g_W}{c\,\rho_B \cos^2\gamma} + \theta_{B\ell}\frac{\partial}{\partial z}\left[0.1464\, R U_\ell \delta_\ell - 0.02723\, U_\ell \delta_\ell^{2}\right] .$$

Introducing values for U_ℓ and δ_ℓ from (19) results in

$$\left[\frac{g_W A^{\frac{2}{3}} \cos^{\frac{1}{4}}\gamma}{0.0228\, c\, \rho_B\, \nu^{\frac{1}{4}}}\right]\frac{\partial}{\partial z}\left[0.03663\, R\, U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{5}{4}}\left(1-\frac{z}{\Delta}\right) - 0.004785\, U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{9}{4}}\left(1-\frac{z}{\Delta}\right)^{\frac{12}{7}}\right]$$

$$+ \frac{\partial}{\partial z}\left[0.1464\, R\,\theta_{B\ell} U_\Delta \delta_\Delta \left(1-\frac{z}{\Delta}\right)^{\frac{8}{7}} - 0.02723\,\theta_{B\ell} U_\Delta \delta_\Delta^{2}\left(1-\frac{z}{\Delta}\right)^{\frac{13}{7}}\right]$$

$$\equiv \frac{R g_W}{c\,\rho_B \cos^2\gamma} + \theta_{B\ell}\frac{\partial}{\partial z}\left[0.1464\, R U_\Delta \delta_\Delta \left(1-\frac{z}{\Delta}\right)^{\frac{8}{7}} - 0.02723\, U_\Delta \delta_\Delta^{2}\left(1-\frac{z}{\Delta}\right)^{\frac{13}{7}}\right] .$$

Performing the indicated differentiations:

$$\left[\frac{g_W A^{\frac{2}{3}} \cos^{\frac{1}{4}}\gamma}{0.0228\, c\, \rho_B\, \nu^{\frac{1}{4}}}\right]\left\{\frac{(12)(0.004785)}{7\Delta}\, U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{9}{4}}\left(1-\frac{z}{\Delta}\right)^{\frac{5}{7}} - 0.03663\, U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{5}{4}}\frac{R}{\Delta}\right.$$

$$\left. + 0.03663\, U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{5}{4}}\left(1-\frac{z}{\Delta}\right)\frac{\partial R}{\partial z}\right\} - \frac{(8)(0.1464)}{7\Delta}\, U_\Delta \delta_\Delta R\,\theta_{B\ell}\left(1-\frac{z}{\Delta}\right)^{\frac{1}{7}}$$

$$+ 0.1464\, U_\Delta \delta_\Delta \left(1-\frac{z}{\Delta}\right)^{\frac{8}{7}} R\frac{\partial \theta_{B\ell}}{\partial z} + 0.1464\, U_\Delta \delta_\Delta \theta_{B\ell}\left(1-\frac{z}{\Delta}\right)^{\frac{8}{7}}\frac{\partial R}{\partial z}$$

$$+ \frac{(13)(0.02723)}{7\Delta}\, U_\Delta \delta_\Delta^{2}\theta_{B\ell}\left(1-\frac{z}{\Delta}\right)^{\frac{6}{7}} - 0.02723\, U_\Delta \delta_\Delta^{2}\left(1-\frac{z}{\Delta}\right)^{\frac{13}{7}}\frac{\partial \theta_{B\ell}}{\partial z}$$

$$\equiv \frac{R g_W}{c\,\rho_B \cos^2\gamma} - \frac{(8)(0.1464)}{7\Delta}\, U_\Delta \delta_\Delta R\,\theta_{B\ell}\left(1-\frac{z}{\Delta}\right)^{\frac{1}{7}}$$

$$+ 0.1464\, U_\Delta \delta_\Delta \theta_{B\ell}\left(1-\frac{z}{\Delta}\right)^{\frac{8}{7}}\frac{\partial R}{\partial z} + \frac{(13)(0.02723)}{7\Delta}\, U_\Delta \delta_\Delta^{2}\theta_{B\ell}\left(1-\frac{z}{\Delta}\right)^{\frac{6}{7}} .$$

Note that the last three terms on the right side have identical terms of the same sign on the left side, thus canceling each other and the equation becomes

$$\frac{\partial \theta_{Bl}}{\partial Z}\left\{0.1464\, U_\Delta \delta_\Delta \left(1-\frac{Z}{\Delta}\right)^{\frac{8}{7}} R - 0.02723\, U_\Delta \delta_\Delta^2 \left(1-\frac{Z}{\Delta}\right)^{\frac{13}{7}}\right\}$$
$$\equiv \frac{R q_W}{c \rho_B \cos^2\gamma} - \left[\frac{q_W Pr^{\frac{2}{3}} \cos^{\frac{1}{4}}\gamma}{0.0228\, c \rho_B \nu^{\frac{1}{4}}}\right]\left\{\frac{(12)(0.004785)}{7\Delta} U_\Delta^{\frac{1}{4}} \delta_\Delta^{\frac{9}{4}} \left(1-\frac{Z}{\Delta}\right)^{\frac{5}{7}}\right. \tag{12f}$$
$$\left. - 0.03663\, U_\Delta^{\frac{1}{4}} \delta_\Delta^{\frac{5}{4}} \frac{R}{\Delta} + 0.03663\, U_\Delta^{\frac{1}{4}} \delta_\Delta^{\frac{5}{4}} \left(1-\frac{Z}{\Delta}\right) \frac{\partial R}{\partial Z}\right\} .$$

Rearranging results in

$$\frac{\partial \theta_{Bl}}{\partial Z}$$
$$\equiv \left[\frac{6.8329\, q_W}{c \rho_B U_\Delta \delta_\Delta \cos^2\gamma}\right]\left\{\frac{1-\left[\frac{1.6066\, Pr^{\frac{2}{3}} U_\Delta^{\frac{1}{4}} \delta_\Delta^{\frac{5}{4}} \cos^{\frac{9}{4}}\gamma}{\nu^{\frac{1}{4}} \Delta}\right]\left[0.2239 \frac{\delta_\Delta}{R}\left(1-\frac{Z}{\Delta}\right)^{\frac{5}{7}} - 1 + \left(1-\frac{Z}{\Delta}\right)\frac{\Delta}{R}\frac{\partial R}{\partial Z}\right]}{\left(1-\frac{Z}{\Delta}\right)^{\frac{8}{7}}\left[1-0.1861\frac{\delta_\Delta}{R}\left(1-\frac{Z}{\Delta}\right)^{\frac{5}{7}}\right]}\right\} . \tag{12g}$$

For a right circular cylinder this reduces to

$$\frac{\partial \theta_{Bl}}{\partial Z} \equiv \left[\frac{6.8329\, q_W}{c \rho_B U_\Delta \delta_\Delta}\right]\left\{\frac{1-\left[\frac{1.6066\, Pr^{\frac{2}{3}} U_\Delta^{\frac{1}{4}} \delta_\Delta^{\frac{5}{4}}}{\nu^{\frac{1}{4}} \Delta}\right]\left[0.2239 \frac{\delta_\Delta}{R_C}\left(1-\frac{Z}{\Delta}\right)^{\frac{5}{7}} - 1\right]}{\left(1-\frac{Z}{\Delta}\right)^{\frac{8}{7}}\left[1-0.1861\frac{\delta_\Delta}{R_C}\left(1-\frac{Z}{\Delta}\right)^{\frac{5}{7}}\right]}\right\} . \tag{12h}$$

The above differential equation can be most readily solved on a digital computer by numerical integration to obtain θ_{Bl} as a function of Z .

Ullage Space Temperature Gradient

Expressions developed previously for boundary layer growth and stratified layer growth rate are valid in the ullage space up to the point at which the stratified layer fully occupies the ullage space. However, temperature gradients in the stratified layer will not be represented accurately by the above developed expression. The assumption of constant wall heat flux used previously will not apply in the region where the wall is wetted by vapor and large temperature increases may occur. Experimental results [Maxson, 1963] demonstrate that large temperature increases can be realized in the ullage space. As a result of these differences a modification of the equation developed for the liquid phase is in order.

Temperature and velocity profiles in the boundary layer are assumed to be of the same form as before. Decay of the boundary layer as it traverses the stratified layer is also assumed to take the same pattern. The momentum equation is developed as previously with identical expressions for the constants and exponents being obtained. Setting up of the energy equation as it pertains to the stratified layer proceeds along parallel lines up to the point where differentiation is to be performed. At that point, the equation is

$$\frac{\partial}{\partial z}\left\{\left[\frac{q_w Pr^{\frac{2}{3}} \cos^{\frac{1}{4}}\gamma}{0.0228\, c_s \rho_s \nu^{\frac{1}{4}}}\right]\left[0.03663\, R\, U_l^{\frac{1}{4}}\delta_l^{\frac{5}{4}} - 0.004785\, U_l^{\frac{1}{4}}\delta_l^{\frac{9}{4}}\right] + \left[0.1464\, R\, \theta_{Bl} U_l \delta_l - 0.02723\, \theta_{Bl} U_l \delta_l^{2}\right]\right\}$$
$$\equiv \frac{R q_w}{c_s \rho_s \cos^2\gamma} + \theta_{Bl}\frac{\partial}{\partial z}\left[0.1464\, R\, U_l \delta_l - 0.02723\, U_l \delta_l^{2}\right] . \qquad (21)$$

Heat transfer through the wall, q_w , is now assumed to be proportional to the temperature difference. Thus

$$q_w = \Omega (T_a - T_{lw}) = \Omega\theta_{al} , \qquad (22)$$

where Ω is the overall coefficient of heat transfer. Now

$$\theta_{al} = \theta_a - \theta_{Bl} .$$

Therefore

$$q_w = \Omega(\theta_a - \theta_{B\ell}) \, .$$

Making this substitution together with the substitution for U_ℓ and δ_ℓ leads to

$$\frac{\partial}{\partial Z}\left\{\left[\frac{R^{\frac{2}{3}}\Omega(\theta_a-\theta_{B\ell})\cos^{\frac{1}{4}}\gamma}{0.0228\, c_s \rho_s\, \nu^{\frac{1}{4}}}\right]\left[0.03663\, R U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{5}{4}}\left(1-\frac{Z}{\Delta}\right)-0.004785\, U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{9}{4}}\left(1-\frac{Z}{\Delta}\right)^{\frac{12}{7}}\right]\right.$$
$$\left.+\left[0.1464\, R\,\theta_{B\ell} U_\Delta \delta_\Delta \left(1-\frac{Z}{\Delta}\right)^{\frac{8}{7}}-0.02723\,\theta_{B\ell} U_\Delta \delta_\Delta^2\left(1-\frac{Z}{\Delta}\right)^{\frac{13}{7}}\right]\right\}$$
$$\equiv \frac{R\Omega(\theta_a-\theta_{B\ell})}{c_s\rho_s \cos^2\gamma}+\theta_{B\ell}\frac{\partial}{\partial Z}\left[0.1464\, R U_\Delta\delta_\Delta\left(1-\frac{Z}{\Delta}\right)^{\frac{8}{7}}-0.02723\, U_\Delta\delta_\Delta^2\left(1-\frac{Z}{\Delta}\right)^{\frac{13}{7}}\right] .$$

After performing the differentiation and collecting terms this becomes

$$\frac{\partial\theta_{B\ell}}{\partial Z}\left\{\left[\frac{R^{\frac{2}{3}}\Omega U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{5}{4}}\cos^{\frac{1}{4}}\gamma}{0.0228\, c_s\rho_s\,\nu^{\frac{1}{4}}}\right]\left[0.004785\frac{\delta_\Delta}{R}\left(1-\frac{Z}{\Delta}\right)^{\frac{12}{7}}-0.03663\left(1-\frac{Z}{\Delta}\right)\right]\right.$$
$$\left.+U_\Delta\delta_\Delta\left[0.1464\left(1-\frac{Z}{\Delta}\right)^{\frac{8}{7}}-0.02723\frac{\delta_\Delta}{R}\left(1-\frac{Z}{\Delta}\right)^{\frac{12}{7}}\right]\right\}$$
$$+(\theta_a-\theta_{B\ell})\left\{\left[\frac{R^{\frac{2}{3}}\Omega U_\Delta^{\frac{1}{4}}\delta_\Delta^{\frac{5}{4}}\cos^{\frac{1}{4}}\gamma}{0.0228\, c_s\rho_s\,\nu^{\frac{1}{4}}\Delta}\right]\left[0.03663\frac{\Delta}{R}\left(1-\frac{Z}{\Delta}\right)\frac{\partial R}{\partial Z}-0.03663\right.\right.$$
$$\left.\left.+\,0.004785\left(\frac{12}{7}\right)\frac{\delta_\Delta}{R}\left(1-\frac{Z}{\Delta}\right)^{\frac{5}{7}}\right]-\frac{\Omega}{c_s\rho_s\cos^2\gamma}\right\}\equiv 0 \, . \qquad (21a)$$

Solution of this equation to obtain $\theta_{B\ell}$ as a function of Z is accomplished by a combined iteration and numerical integration computer program. Below the stratified layer (i. e., $Z < 0$) the vapor temperature is that of saturated gas, i. e., $\theta_{B\ell} = 0$.

Self Pressurization Computation

Having obtained, for a particular time, the temperature pattern in the ullage space a pressure is now determined for the ullage space that is compatible with the temperature variation and the mass present in the ullage. To accomplish this a differential volume element described by

$$dV = \pi R^2 d\psi$$

is used.

Note that the variable ψ introduced here is a variable over the entire height of the ullage space (including the above named Z) regardless of whether the ullage stratified layer has occupied the entire height (figure 1).

The mass contained in this element is then given by

$$dM = \pi \rho R^2 d\psi \ ,$$

and the total mass by

$$M = \pi \int_0^h \rho R^2 d\psi \ .$$

ρ is a function of the local temperature and pressure which is symbolized as

$$\rho = \rho(\theta_{B\ell}, P_S)$$

while R is a function of ψ and is given by

$$R = R_S - \psi \tan \gamma \ .$$

Since the previous determination gave $\theta_{B\ell}$ as a function of Z, and therefore of ψ, where ψ includes the region below the stratified layer as well as the layer where a temperature gradient exists ($0 < Z < \Delta$), the total mass may be determined from

$$M = \pi \int_0^h \rho(\theta_{B\ell}, P_S)\left[R_S - \psi \tan \gamma\right]^2 d\psi \ . \tag{23a}$$

where a combined iteration and numerical integration procedure is again used. The expression is numerically integrated using various assumed pressures until the computed mass compares with the known mass from initial conditions or previous steps. The pressure obtained from this procedure has a matching saturation temperature which is compared with the liquid surface temperature obtained from the thermal analysis of the stratified liquid layer. If a satisfactory match is not made, then vaporization of liquid or condensation of gas is permitted to bring the liquid surface temperature into equilibrium with the computed pressure.

Vaporization will occur if the liquid surface temperature obtained from (11) exceeds the saturation temperature determined above. A liquid surface layer of thickness Z_l at a constant temperature equal to the saturation temperature is assumed as shown in figure 4a. Thermal energy proportional to the shaded area is removed from the liquid by vaporization and the mass of liquid vaporized computed from

$$DM_V = \frac{\pi}{L_V} \int_0^{Z_l} c_s \rho_s (\theta_{B\ell} - \theta_s) R^2 dz_l \quad . \tag{24}$$

When the liquid surface temperature is less than saturation temperature condensation is allowed until the liquid temperature is raised to the saturation point. The temperature pattern assumed is shown in figure 4b. An error function expression [Schmidt, et al., 1961] is assumed to represent the temperature variation as though the surface temperature had experienced a step change in temperature from T_B to T_S. The equation is given as

$$\frac{\theta_s - \theta_{B\ell}}{\theta_s} = \frac{2}{\sqrt{\pi}} \int_0^{\phi} e^{-\lambda^2} d\lambda \quad , \tag{25}$$

where $\phi = \frac{Z_1}{2\sqrt{\alpha t}}$ and t is taken as the time interval used in (18a). An iteration procedure is used to determine a Z_1 where the θ_{Bl} matches the θ_{Bl} from (12g). Equation (24) is again used to determine the mass transport, however, the term ($\theta_{Bl_{max}} - \theta_S$) will now be negative signifying removal of mass from the ullage space.

Use of Equations

Combining of the above developed equations to provide a quantitative solution for a tank self pressurization problem proceeds as outlined below.

The depth of the stratified layer, Δ, is determined from (18a). A numerical integration of (18a) is made using a selected depth increment, $D\Delta$, to determine the corresponding time increment, Dt. A tabulation or graph of $\sum D\Delta$ versus $\sum Dt$ suffices to establish the variation of Δ with t.

In order to accomplish the above integration certain preliminary determinations must be made. U_Δ and δ_Δ may be computed from (6) using the values of m = 3/7, n = 5/7 and (13a) and (14a) to determine the constants. Changes in tank shape or heat transfer with height may be taken care of by repetitive use of equations such as (16) and (17). R_Δ as well as U_Δ and δ_Δ , must be evaluated at the bottom of the stratified layer.

Once the vertical growth with time of the stratified layer is obtained the temperature gradient within the stratified layer may be determined using (12g). Again the terms U_Δ , δ_Δ , and R_Δ are evaluated at the bottom of the stratified layer.

In all of the above calculations the fluid properties are calculated at bulk liquid conditions.

Parallel calculations to establish stratified layer depth as a function of time and the temperature gradient in the ullage space must be accomplished at this point. An identical procedure to that described for the liquid phase is used to establish stratified layer depth with fluid properties evaluated at saturated gas conditions. Equations (6), (13a), (14a), and (18a) are used for this determination. The temperature gradient through the gas stratified layer is computed by a combined numerical integration and iteration solution of (21a).

Having established a temperature pattern in the ullage space, an iteration procedure is used to establish the ullage space pressure from (23a). The mass calculated from (23a) is forced to be equal to the initial mass in the ullage volume by an iterative adjustment of pressure. Temperature below the stratified layer in the ullage is assumed to be that of saturation. The saturation temperature corresponding to the above determined pressure is then compared with the liquid surface temperature determined previously. Mass interchange is determined for either condensation or vaporization and the mass vaporized determined from (24) is algebrically added to the ullage mass present at the start of the time interval. This new mass is now used for the next time interval.

Repetitive application of the above procedure for successive time increments will enable a determination of ullage pressure as a function of time. In the interface calculations the temperature profile in the liquid and the pressure in the ullage space must be determined for identical times.

Boundary layer behavior and temperature gradient in the ullage space must be computed from the beginning for each time increment due to the change in fluid properties which may occur as a result of pressure and temperature differences.

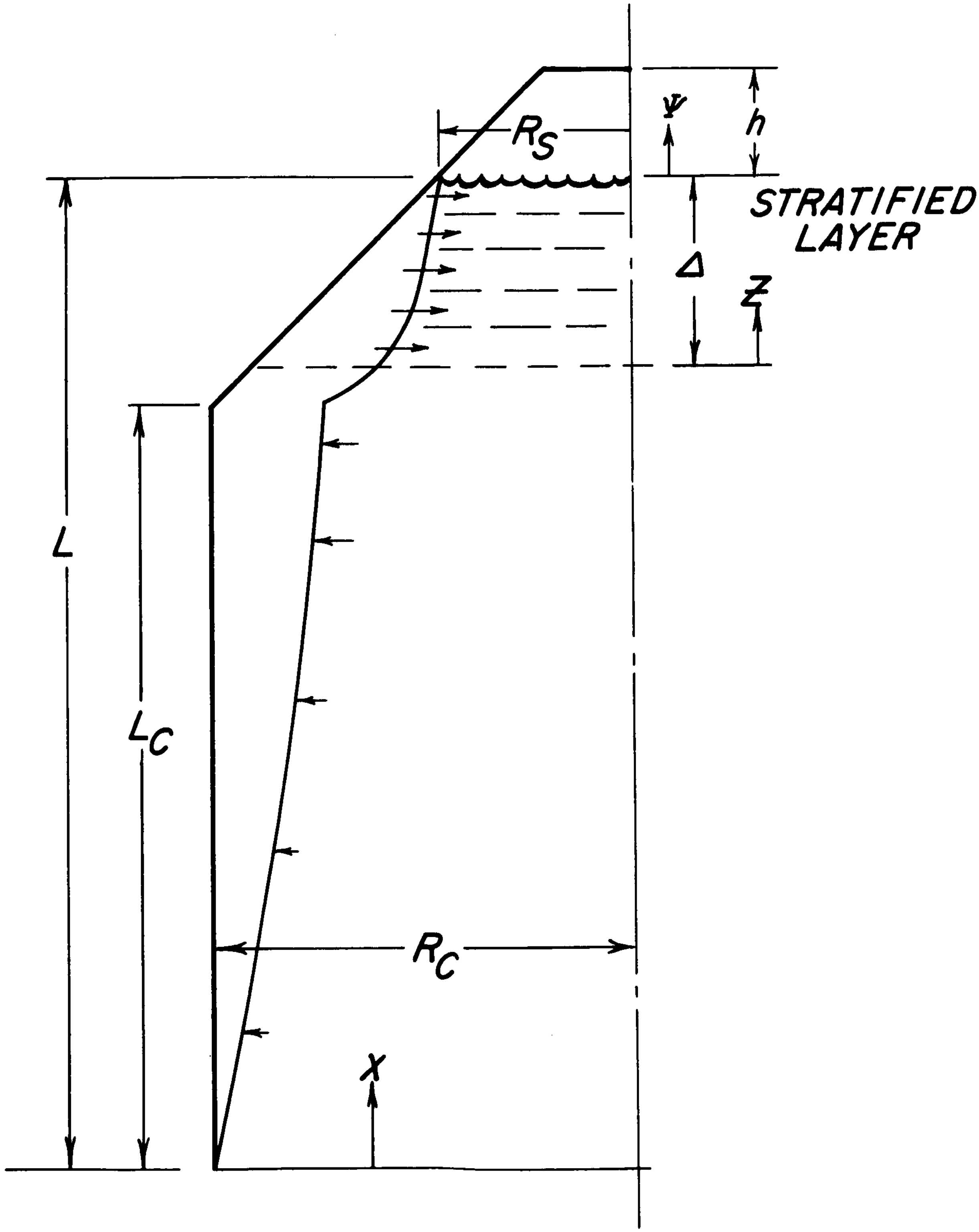


Figure 1. Assumed Geometry

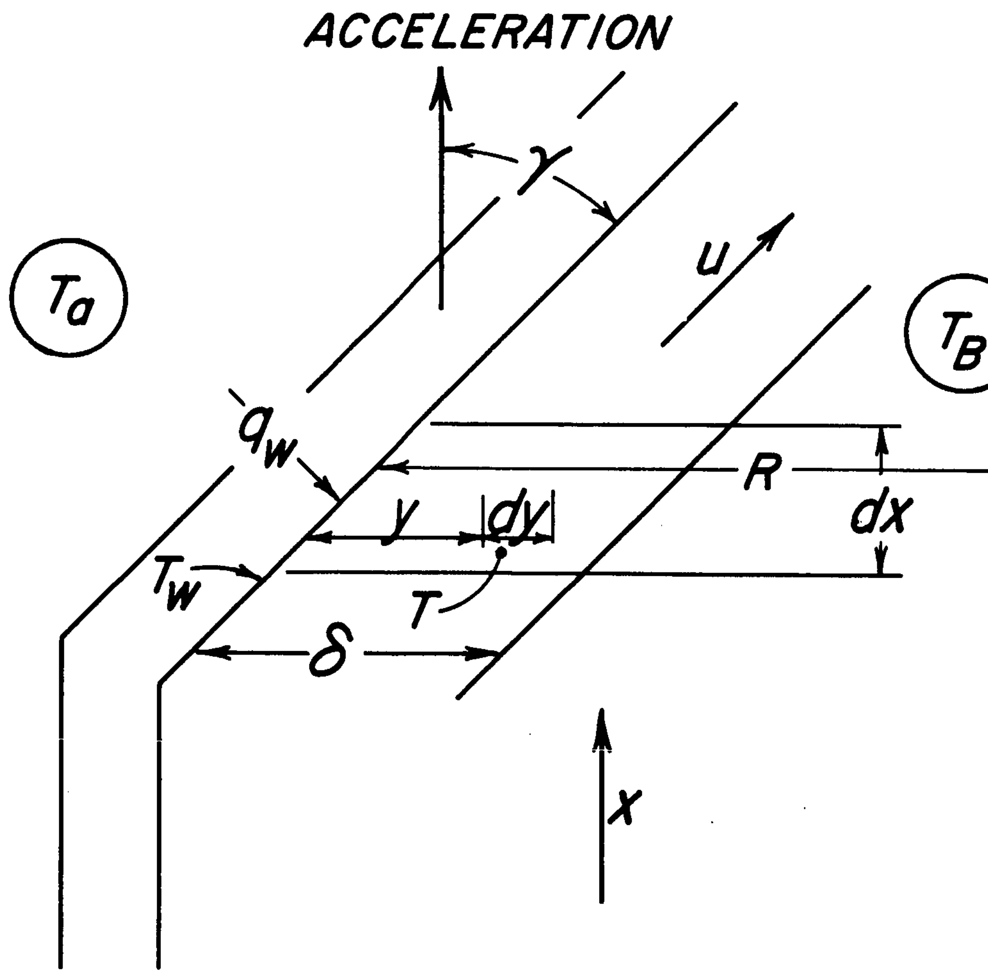


Figure 2. Element Considered For Boundary Layer Growth

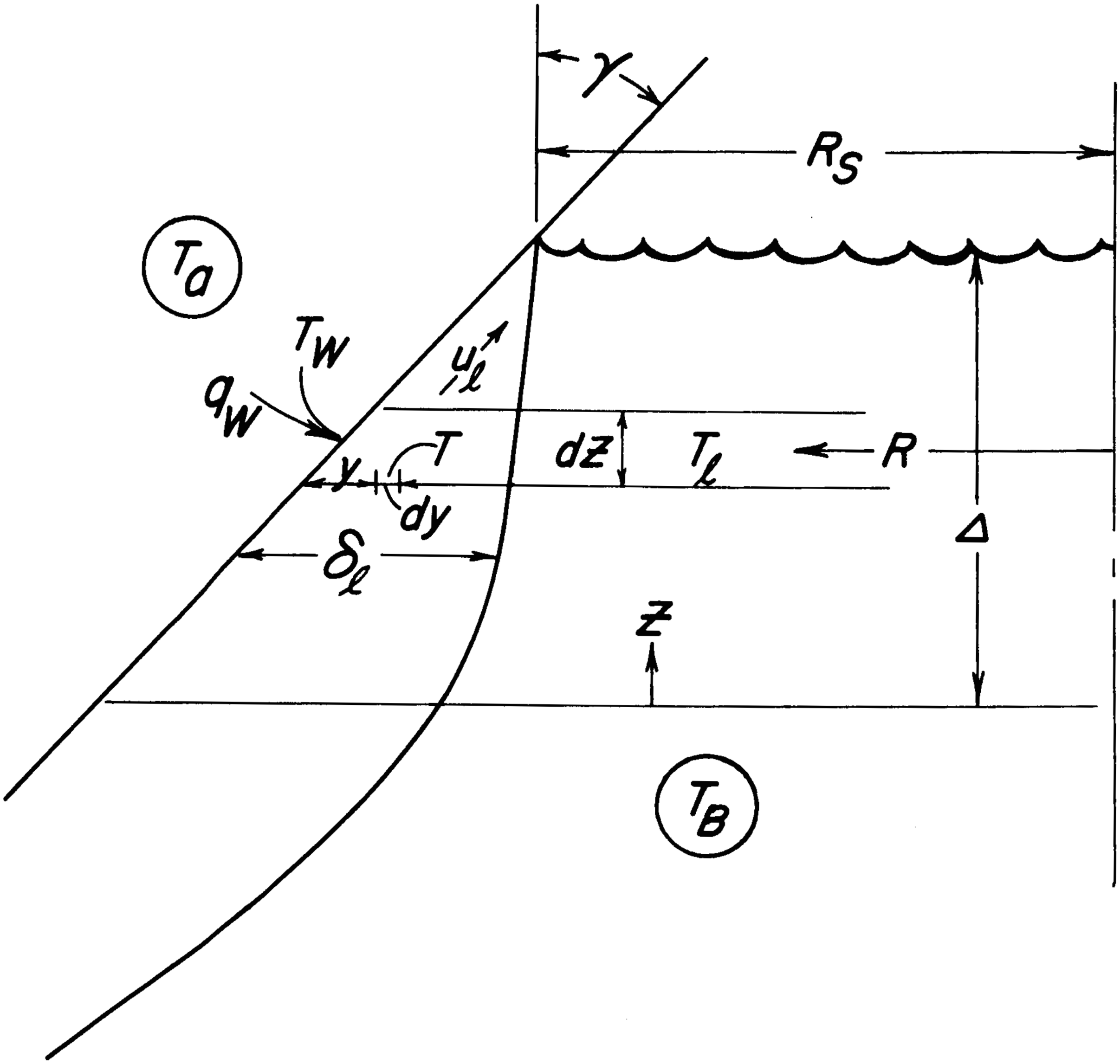
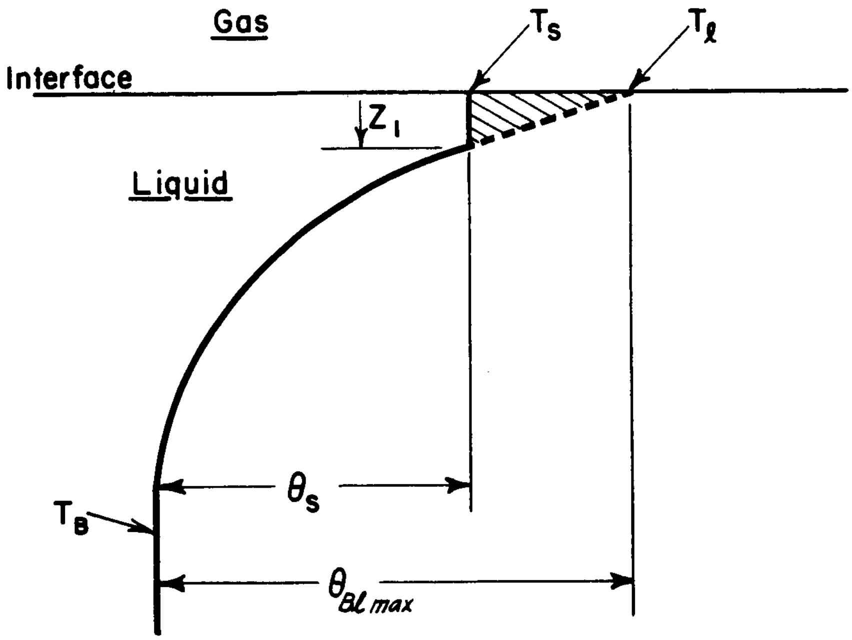
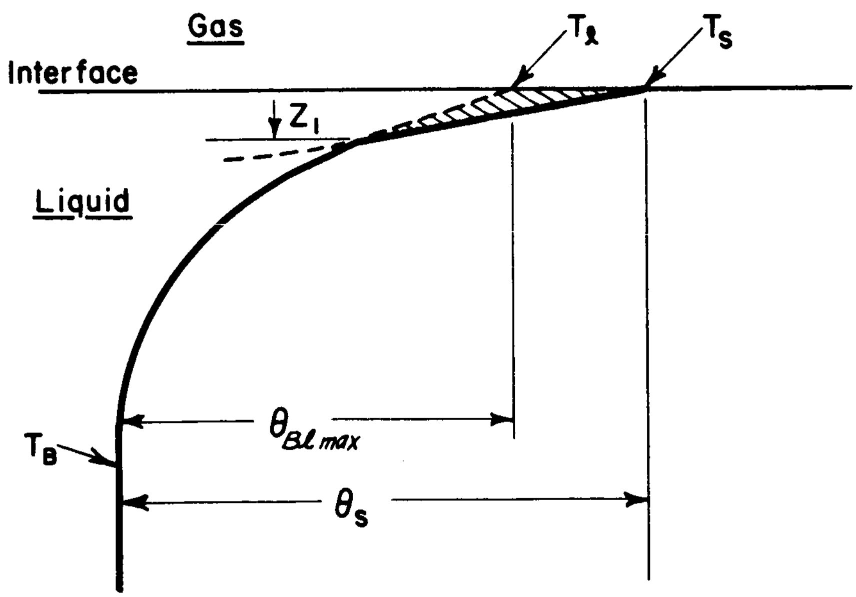


Figure 3. Element Considered For Boundary Layer Decay



(a) Evaporation



(b) Condensation

Figure 4. Interface Temperature Adjustment

Nomenclature

A - cross section area

C_1, C_2, F, m, n, r, p - derived constants

c - specific heat

G_{RL}^{*} - modified Grashof number, $= \dfrac{g \beta q_w L^4}{k \nu^2}$

g - acceleration

h - vapor space height

K - substitutional term

k - thermal conductivity

L - liquid depth

L_c - length of cylindrical tank

M - mass

P - pressure

Pr - Prandtl number

q - unit area heat flux

R - tank radius

R_c - cylinder radius

R_s - radius at the liquid surface

T - absolute temperature

t - time

u - local velocity in boundary layer

U - equivalent free stream velocity

V - volume

x - vertical distance in tank

y - horizontal distance from tank wall

Z - vertical distance from bottom of stratified layer

a - liquid diffusivity

β - volumetric coefficient of thermal expansion

γ - half angle of conical nose

Nomenclature (continued)

Δ - stratified layer thickness

δ - boundary layer thickness

λ - error function variable

θ - temperature difference, $T - T_B$.

$\theta_a = T_a - T_B$

$\theta_{a\ell} = T_a - T_\ell$

$\theta_{B\ell} = T_\ell - T_B$

$\theta_W = T_W - T_B$

$\theta_\ell = T - T_\ell$

$\theta_S = T_S - T_B$

μ - dynamic viscosity

ν - kinematic viscosity

ρ - fluid mass density

τ - viscous shear stress

ψ - vertical distance in ullage space

Ω - overall tank wall heat transfer coefficient

Subscripts

a - ambient conditions (external to tank)

B - at bulk fluid condition

c - in the cylindrical portion

ℓ - in stratified layer

n - in the conical nose

u - in the ullage space

W - at tank wall

S - saturated conditions

Δ - at bottom of stratified layer

δ - pertaining to boundary layer

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Appendix A

Evaluation of Integrals

1. In the development of the preceding equations several integrals need to be evaluated. Following are the detailed integral evaluations.

Each integral evaluated involves the term y/δ. For ease in handling the integrals the substitution $y/\delta = a$ was used. Thus $\delta a = y$, and $\delta\, da = dy$. To establish limits it is observed that when $y = o$, $a = o$; when $y = \delta$, $a = 1$.

In all of the integrals the relations

$$\theta = \theta_w \left[1 - \left(\frac{y}{\delta}\right)^{1/7} \right] = \theta_w \left(1 - a^{1/7} \right),$$

and

$$u = U\left(\frac{y}{\delta}\right)^{1/7} \left(1 - \frac{y}{\delta}\right)^4 = U a^{1/7} (1 - a)^4$$

are used.

2. Evaluation of integrals.

(a) $$\int_o^{\delta} u^2\, dy = \int_o^1 \delta U^2 a^{2/7} (1 - a)^8\, da = \delta U^2 \int_o^1 a^{2/7} (1 - a)^8\, da$$

$$= \delta U^2 \int_o^1 \left[a^{2/7} - 8a^{9/7} + 28a^{16/7} - 56a^{23/7} + 70a^{30/7} - 56a^{37/7} \right.$$

$$\left. + 28a^{44/7} - 8a^{51/7} + a^{58/7} \right] da$$

$$\int_0^\delta u^2\,dy = \delta U^2(7)\left[\frac{1}{9}a^{9/7} - \frac{1}{2}a^{16/7} + \frac{28}{23}a^{23/7} - \frac{56}{30}a^{30/7} + \frac{70}{37}a^{37/7}\right.$$

$$\left. - \frac{14}{11}a^{44/7} + \frac{28}{51}a^{51/7} - \frac{4}{29}a^{58/7} + \frac{1}{65}a^{65/7}\right]_0^1$$

$$= \delta U^2(7)\left[\left(\frac{1}{9} + \frac{28}{23} + \frac{70}{37} + \frac{28}{51} + \frac{1}{65}\right) - \left(\frac{1}{2} + \frac{28}{15} + \frac{14}{11} + \frac{4}{29}\right)\right]$$

$$= \delta U^2(7)\,(3.7847985 - 3.7773250) = \underline{0.052315\,\delta\,U^2}.$$

(b) $$\int_0^\delta u^2 y\,dy = \int_0^1 \delta^2 U^2 a^{9/7}(1-a)^8\,da = \delta^2 U^2 \int_0^1 a^{9/7}(1-a)^8\,da$$

$$= \delta^2 U^2 \int_0^1 \left[a^{9/7} - 8a^{16/7} + 28a^{23/7} - 56a^{30/7} + 70a^{37/7} - 56a^{44/7}\right.$$

$$\left. + 28a^{51/7} - 8a^{58/7} + a^{65/7}\right] da$$

$$= \delta^2 U^2(7)\left[\frac{1}{16}a^{16/7} - \frac{8}{23}a^{23/7} + \frac{14}{15}a^{30/7} - \frac{56}{37}a^{37/7} + \frac{35}{22}a^{44/7}\right.$$

$$\left. - \frac{56}{51}a^{51/7} + \frac{14}{29}a^{58/7} - \frac{8}{65}a^{65/7} + \frac{1}{72}a^{72/7}\right]_0^1$$

$$= \delta^2 U^2(7)\left[\left(\frac{1}{16} + \frac{14}{15} + \frac{35}{22} + \frac{14}{29} + \frac{1}{72}\right) - \left(\frac{8}{23} + \frac{56}{37} + \frac{56}{51} + \frac{8}{65}\right)\right]$$

$$= \delta^2 U^2(7)\,(3.08338993 - 3.08245574) = \underline{0.0065393\,\delta^2 U^2}.$$

(c) $$\int_0^\delta \theta\,dy = \int_0^1 \delta\,\theta_w(1 - a^{1/7})da = \delta\,\theta_w \int_0^1 (1 - a^{1/7})da$$

$$= \delta\theta_w \left[a - \frac{7}{8}a^{8/7}\right]_0^1 = \delta\,\theta_w\left[1 - \frac{7}{8}\right] = \underline{0.125\,\delta\,\theta_w}.$$

(d) $$\int_0^\delta \theta y\,dy = \int_0^1 \delta^2\theta_w\,a(1 - a^{1/7})da = \delta^2\theta_w \int_0^1 a(1 - a^{1/7})da$$

$$= \delta^2\theta_w\left[\frac{1}{2}a^2 - \frac{7}{15}a^{15/7}\right]_0^1 = \delta^2\theta_w\left(\frac{1}{2} - \frac{7}{15}\right) = \underline{\frac{1}{30}\,\delta^2\,\theta_w}.$$

(e) $$\int_0^\delta \theta\,u\,dy = \int_0^1 \theta_w\,U\,\delta a^{1/7}\,(1 - a)^4\,(1 - a^{1/7})\,da$$

$$= \theta_w U\,\delta \int_0^1 a^{1/7}(1 - a)^4\,(1 - a^{1/7})\,da$$

$$= \theta_w U\,\delta \int_0^1 \Big[a^{1/7} - a^{2/7} - 4a^{8/7} + 4a^{9/7} + 6a^{15/7} - 6a^{16/7} - 4a^{22/7}$$

$$+ 4a^{23/7} + a^{29/7} - a^{30/7}\Big]da$$

$$= \theta_w U\,\delta\,(7)\Big[\frac{1}{8}a^{8/7} - \frac{1}{9}a^{9/7} - \frac{4}{15}a^{15/7} + \frac{1}{4}a^{16/7} + \frac{3}{11}a^{22/7} - \frac{6}{23}a^{23/7}$$

$$- \frac{4}{29}a^{29/7} + \frac{2}{15}a^{30/7} + \frac{1}{36}a^{36/7} - \frac{1}{37}a^{37/7}\Big]_0^1$$

$$\int_o^\delta \theta\, u\, dy = \theta_w U \delta (7) \left[\left(\frac{1}{8} + \frac{1}{4} + \frac{3}{11} + \frac{2}{15} + \frac{1}{36} \right) - \left(\frac{1}{9} + \frac{4}{15} + \frac{6}{23} + \frac{4}{29} + \frac{1}{37} \right) \right]$$

$$= \theta_w U \delta (7)\, (0.8088384 - 0.8036054)$$

$$= \underline{0.036631\, \theta_w U \delta}.$$

(f) $$\int_o^\delta \theta u y\, dy = \int_o^1 \theta_w U \delta^2\, a^{8/7}(1 - a^{1/7})\,(1 - a)^4\, da$$

$$= \theta_w U \delta^2 \int_o^1 a^{8/7}(1 - a^{1/7})\,(1 - a)^4\, da$$

$$= \theta_w U \delta^2 \int_o^1 \left[a^{8/7} - a^{9/7} - 4a^{15/7} + 4a^{16/7} + 6a^{22/7} - 6a^{23/7} - 4a^{29/7} + 4a^{30/7} + a^{36/7} - a^{37/7} \right] da$$

$$= \theta_w U \delta^2 (7) \left[\frac{1}{15} a^{15/7} - \frac{1}{16} a^{16/7} - \frac{2}{11} a^{22/7} + \frac{4}{23} a^{23/7} + \frac{6}{29} a^{29/7} - \frac{1}{5} a^{30/7} - \frac{1}{9} a^{36/7} + \frac{4}{37} a^{37/7} + \frac{1}{43} a^{43/7} - \frac{1}{44} a^{44/7} \right]_o^1$$

$$= \theta_w U \delta^2 (7) \left[\left(\frac{1}{15} + \frac{4}{23} + \frac{6}{29} + \frac{4}{37} + \frac{1}{43} \right) - \left(\frac{1}{16} + \frac{2}{11} + \frac{1}{5} + \frac{1}{9} + \frac{1}{44} \right) \right]$$

$$\int_0^{\delta} \theta u y\, dy = \theta_w U \delta^2 (7)\,(0.57884018 - 0.57815656)$$

$$= \underline{0.0047853\, \theta_w U \delta^2}.$$

(g) $$\int_0^{\delta} u dy = \int_0^1 \delta U a^{1/7}(1 - a)^4\, da = \delta U \int_0^1 a^{1/7}(1 - a)^4\, da$$

$$= \delta U \int_0^1 \left[a^{1/7} - 4a^{8/7} + 6a^{15/7} - 4a^{22/7} + a^{29/7} \right] da$$

$$= \delta U(7) \left[\frac{1}{8} a^{8/7} - \frac{4}{15} a^{15/7} + \frac{3}{11} a^{22/7} - \frac{4}{29} a^{29/7} + \frac{1}{36} a^{36/7} \right]_0^1$$

$$= \delta U(7) \left[\left(\frac{1}{8} + \frac{3}{11} + \frac{1}{36} \right) - \left(\frac{4}{15} + \frac{4}{29} \right) \right]$$

$$= \delta U(7)\,(0.42550505 - 0.40459770)$$

$$= \underline{0.14635\, \delta U}.$$

(h) $$\int_0^{\delta} uy\, dy = \int_0^1 \delta^2 U\, a^{8/7} (1 - a)^4 da = \delta^2 U \int_0^1 a^{8/7} (1 - a)^4 da$$

$$= \delta^2 U \int_0^1 \left[a^{8/7} - 4a^{15/7} + 6a^{22/7} - 4a^{29/7} + a^{36/7} \right] da$$

$$= \delta^2 U(7) \left[\frac{1}{15} a^{15/7} - \frac{2}{11} a^{22/7} + \frac{6}{29} a^{29/7} - \frac{1}{9} a^{36/7} + \frac{1}{43} a^{36/7} \right]_0^1$$

$$= \delta^2 U(7) \left[\left(\frac{1}{15} + \frac{6}{29} + \frac{1}{43} \right) - \left(\frac{2}{11} + \frac{1}{9} \right) \right]$$

$$= \delta^2 U(7)\,(0.29681903 - 0.29292929) = \underline{0.027228\, \delta^2\, U.}$$

Appendix B

Use of (6a) to calculate boundary layer thickness requires use of (13a) and (14a) to determine values for C_1 and C_2. Where varying conditions exist along the vertical height of the vessel, the values of C_1 and C_2 may also change. If the vertical height is divided into increments and the divisions between these increments designated as χ_1, χ_2, $\cdots$ χ_n, then an adjustment can be made to assure continuity of boundary layer thickness and velocity.

Handling of the expressions will be simplified by setting

$$C_1 = C_u$$

and

$$C_2 = C_\delta .$$

Then in general

$$U = C_u \chi^{3/7}$$

and

$$\delta = C_\delta \chi^{5/7} .$$

Now

$$U = C_{uo} \chi^{3/7} , \quad O < \chi \leq \chi_1 .$$

Let

$$U = C_{u1} \chi^{3/7} + F_{u1} , \quad \chi_1 \leq \chi \leq \chi_2 .$$

Then

$$C_{uo} \chi_1^{3/7} = C_{u1} \chi_1^{3/7} + F_{u1} , \text{ and } F_{u1} = C_{u1} \chi_1^{3/7} \left(\frac{C_{uo}}{C_{u1}} - 1 \right) .$$

Also

$$U = C_{u2}\chi^{3/7} + F_{u2}, \qquad \chi_2 \le \chi \le \chi_3.$$

Therefore

$$C_{u1}\chi_2^{3/7} + F_{u1} = C_{u2}\chi_2^{3/7} + F_{u2} \quad \text{and} \quad F_{u2} = C_{u2}\chi_2^{3/7}\left(\frac{C_{u1}}{C_{u2}} - 1\right) + F_{u1}.$$

Generally then

$$U = C_{un}\chi^{3/7} + F_{un}, \qquad \chi_n \le \chi \le \chi_{n+1}$$

where

$$F_{un} = C_{un}\chi_n^{3/7}\left(\frac{C_{u(n-1)}}{C_{un}} - 1\right) + F_{u(n-1)}.$$

Likewise for boundary layer thickness the general expression becomes

$$\delta = C_{\delta n}\chi^{5/7} + F_{\delta n}, \qquad \chi_n \le \chi \le \chi_{n+1}$$

where

$$F_{\delta n} = C_{\delta n}\chi_n^{5/7}\left(\frac{C_{\delta(n-1)}}{C_{\delta n}} - 1\right) + F_{\delta(n-1)}.$$

The above procedure may be used where changes in wall heat transfer or inclination of the surface can be approximated by step changes in values.